Handout: Channel Flow - Mean Velocity Profiles

Similarity

$$\langle U \rangle = f(\rho, \nu, \delta, y, \frac{dp_w}{dx}) = f(\rho, \nu, \delta, y, u_\tau)$$

$\Rightarrow 6 - 3 = 3$ non-dimensional groups

$$\frac{\langle U \rangle}{u_\tau} = f(\frac{y}{\delta}, \frac{\delta}{\delta_\nu}) = f\left(\frac{y}{\delta}, \mathrm{Re}_\tau\right)$$

One can do the same analysis for the Velocity Gradient:

$$\frac{d\langle U \rangle}{dy} = \frac{u_\tau}{y} \Phi(\frac{y}{\delta}, \frac{y}{\delta_\nu})$$

δ_ν is the appropriate length scale in the viscous wall region, while δ is the appropriate length scale in the outer layer

The law of the wall

Prandtl (1925): In the inner layer (y/δ <<1) and at high Reynolds numbers <U> is determined only by the viscous scales

$$\Rightarrow \frac{d\langle U \rangle}{dy} = \frac{u_\tau}{y} \Phi_I(\frac{y}{\delta_\nu}) \quad \text{for } y/\delta << 1 \qquad (1)$$

with

$$u^+ = \frac{\langle U \rangle}{u_\tau} \text{ and } y^+ = \frac{y}{\delta_\nu}$$

$$\Rightarrow \frac{du^+}{dy^+} = \frac{1}{y^+} \Phi_I(y^+) \qquad (2)$$

The integral of the previous equation is the **Law of the Wall** valid in viscous sublayer and the inner layer (y/d<0.1)

$$u^+ = f_w(y^+) \tag{3}$$

where

$$f_w(y^+) = \int_0^{y^+} \frac{1}{y^{+'}} \Phi_I(y^{+'}) dy^{+'}$$

Viscous Sublayer

From equation (3)

$$f_w = u^+ = \frac{\langle U \rangle}{u_\tau}$$

$$\Rightarrow f_w(0) = 0$$

$$f_w^{'}(0) = \left. \frac{du^+}{dy^+} \right|_{y=0}$$

$$= \frac{\delta_\nu}{u_\tau} \frac{\partial \langle U \rangle}{\partial y} |_{y=0}$$

$$= \frac{\nu}{u_\tau^2} \frac{\tau_w}{\rho \nu}$$

$$= \frac{\rho}{\tau_w} \frac{\tau_w}{\rho} = 1$$

Hence, the Taylor series expansion for small y^+

$$u^+ = f_w(y^+) = f_w(0) + f_w^{'}(0) y^+ + O(y^{+2})$$

$$\Rightarrow u^+ = y^+$$

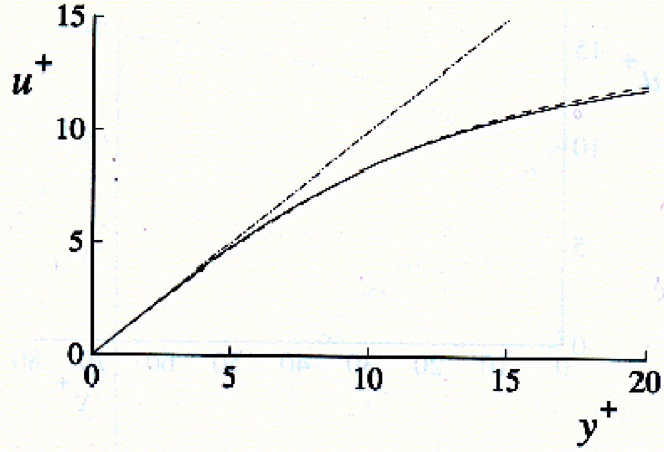

Fig. 7.5. Near-wall profiles of mean velocity from the DNS data of Kim *et al.* (1987): dashed line, Re = 5,600; solid line, Re = 13,750; dot–dashed line, $u^+ = y^+$.

The Log Law

At large Reynolds numbers, the outer part of the inner layer corresponds to large y^+ where viscosity has little effect (Φ independent of $\frac{y}{\delta_\nu}$)

$$\Phi = \Phi(\frac{y}{\delta_\nu}, \frac{y}{\delta})$$

$$y^+ >> 1 \Rightarrow \frac{y}{\delta_\nu} = \frac{y}{\delta}\frac{\delta}{\delta_\nu} >> 1$$

$$\Rightarrow \frac{y}{\delta} >> \frac{\delta_\nu}{\delta} = \frac{\nu}{u_\tau \delta} = \mathrm{Re}_\tau^{-1}$$

Also Φ independent of $\frac{y}{\delta}$ for

$$\frac{y}{\delta} << 1 \quad let's\ say \quad \frac{y}{\delta} < 0.1$$

$$\Rightarrow \frac{y}{\delta_\nu}\frac{\delta_\nu}{\delta} < 0.1$$

$$\Rightarrow y^+ < 0.1\frac{\delta}{\delta_\nu} = 0.1\frac{\delta u_\tau}{\nu} = 0.1\mathrm{Re}_\tau$$

where the edge of the inner layer was defined at 0.1δ.

So if Re_τ is such that $\frac{y}{\delta} >> \mathrm{Re}_\tau^{-1}$ and $y^+ = y/\delta_v < 0.1\mathrm{Re}_\tau$

$\Phi_I(y^+)$ independent of both y/δ_v (viscosity) and of y/δ, thus from equation (2):

$$\Rightarrow \Phi_I = const = \frac{1}{\kappa}$$

$$\Rightarrow \frac{du^+}{dy^+} = \frac{1}{\kappa y^+}$$

$$\Rightarrow u^+ = \frac{1}{\kappa}\ln y^+ + B$$

with

$\kappa = 0.41$ (Von Karman constant)

B=5.2

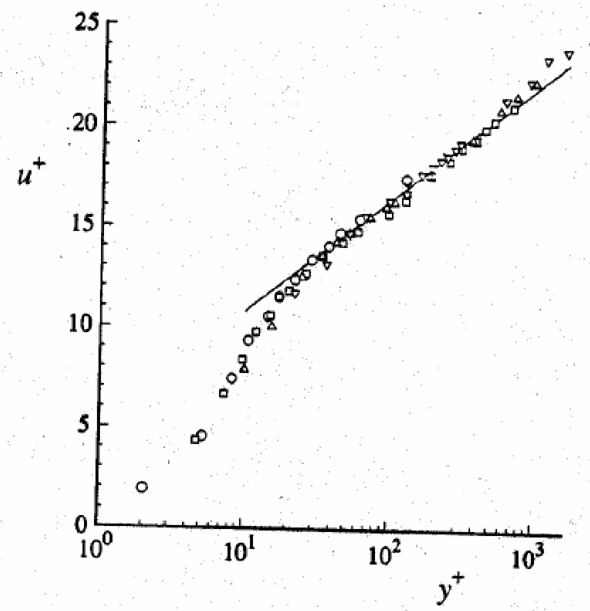

Fig. 7.7. Mean velocity profiles in fully developed turbulent channel flow measured by Wei and Willmarth (1989): ○, $\mathrm{Re}_0 = 2,970$; □, $\mathrm{Re}_0 = 14,914$; Δ, $\mathrm{Re}_0 = 22,776$; ∇, $\mathrm{Re}_0 = 39,582$; line, the log law, Eqs. (7.43)–(7.44).

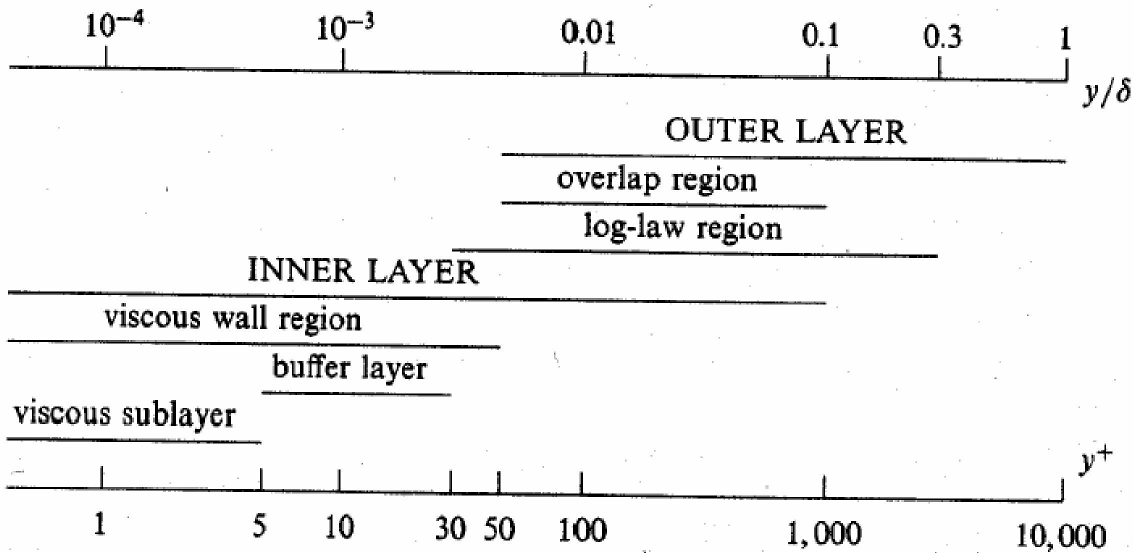

Fig. 7.8. A sketch showing the various wall regions and layers defined in terms of $y^+ = y/\delta_\nu$ and y/δ, for turbulent channel flow at high Reynolds number ($\mathrm{Re}_\tau = 10^4$).

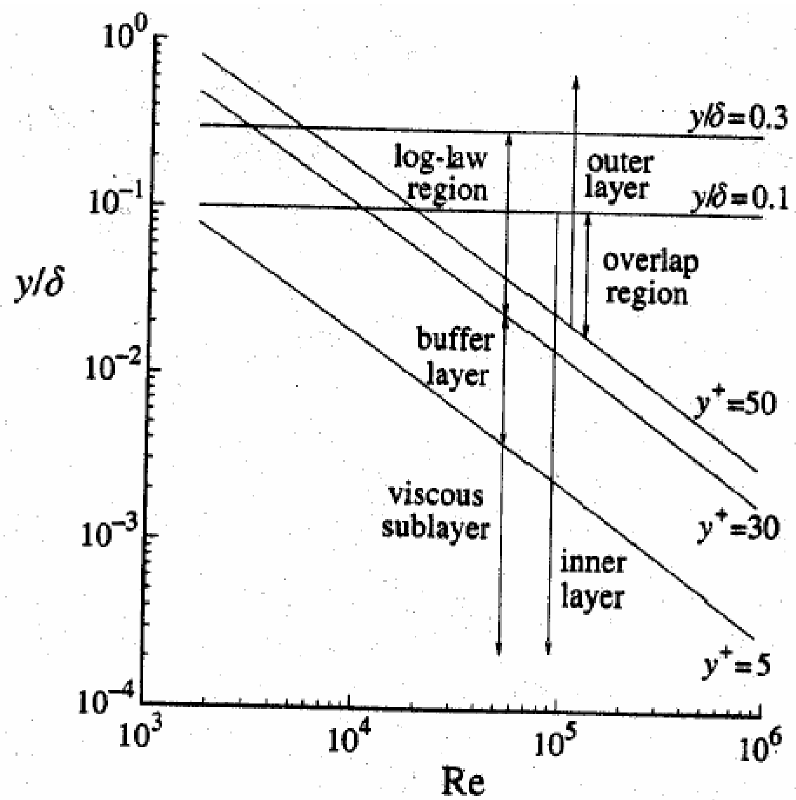

Fig. 7.13. Regions and layers in turbulent channel flow as functions of the Reynolds number.

Velocity Defect Law

In the outer layer $(y^{+} > 50)$ the assumption that Φ is independent of ν implies that, for large y/δ_{ν}, Φ tends asymptotically to a function of y/δ only:

$$\frac{d\langle U\rangle}{dy} = \frac{u_{\tau}}{y}\Phi_{o}(\frac{y}{\delta}) \qquad (4)$$

By integrating in y from y to δ (channel centerline)

$$\int_{y}^{\delta}\frac{d\langle U\rangle}{dy^{'}}dy^{'} = \int_{y}^{\delta}\frac{u_{\tau}}{y^{'}}\Phi o(\frac{y^{'}}{\delta})dy^{'}$$

$$\frac{U_{o}-\langle U\rangle}{u_{\tau}} = F_{D}(\frac{y}{\delta}) \quad \text{with} \qquad F_{D}(\frac{y}{\delta}) = \int_{y/\delta}^{1}\frac{1}{y^{'}}\Phi o(y')dy^{'} \qquad (5)$$

Unlike the law of the wall function $f_{w}(y^{+})$ which is universal, the function $F_{D}(\frac{y}{\delta})$ is different in different flows.

At sufficiently high Reynolds numbers (approximately Re>20,000) there is an overlap region between the inner layer and the outer layer (see Figs. 7.8 and 7.13). In this region both equations (1) and (4) are valid

$$\frac{y}{u_{\tau}}\frac{d\langle U\rangle}{dy} = \Phi_{I}\left(\frac{y}{\delta_{\nu}}\right) = \Phi_{0}\left(\frac{y}{\delta}\right) \quad \text{for} \quad \delta_{\nu} << y << \delta$$

This can be satisfied only if the two functions on the right hand side are equal to the same constant (1/κ), which leads to

$$\Rightarrow \frac{d\langle U\rangle}{dy} = \frac{u_{\tau}}{y\kappa} \text{ for } \delta_{\nu} << y << \delta$$

For small y/δ, the velocity defect law (equation 5) can be written as:

$$\frac{U_{0}-\langle U\rangle}{u_{\tau}} = F_{D}\left(\frac{y}{\delta}\right) = -\frac{1}{\kappa}\ln\frac{y}{\delta} + B_{1} \qquad (6)$$

The value of the flow-dependent constant B_1 is ~0.2 based on DNS and ~0.7 based on experimental data for channel flows.

Statistics in turbulent channel flows:

Table 7.2. *Statistics in turbulent channel flow, obtained from the DNS data of Kim* et al. *(1987), Re = 13,750*

	Location		
	Peak production $y^+ = 11.8$	Log law $y^+ = 98$	Centerline $y^+ = 395$
$\langle u^2\rangle/k$	1.70	1.02	0.84
$\langle v^2\rangle/k$	0.04	0.39	0.57
$\langle w^2\rangle/k$	0.26	0.59	0.59
$\langle uv\rangle/k$	−0.116	−0.285	0
ρ_{uv}	−0.44	−0.45	0
$\mathcal{S}k/\varepsilon$	15.6	3.2	0
$\mathcal{P}/\varepsilon$	1.81	0.91	0

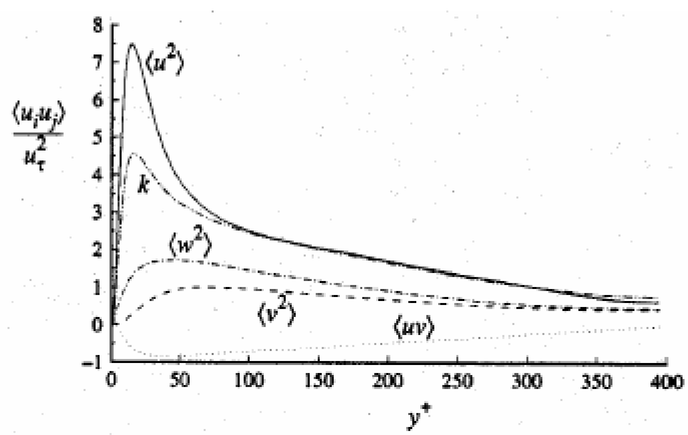

Fig. 7.14. Reynolds stresses and kinetic energy normalized by the friction velocity against y^+ from DNS of channel flow at Re = 13,750 (Kim *et al.* 1987).

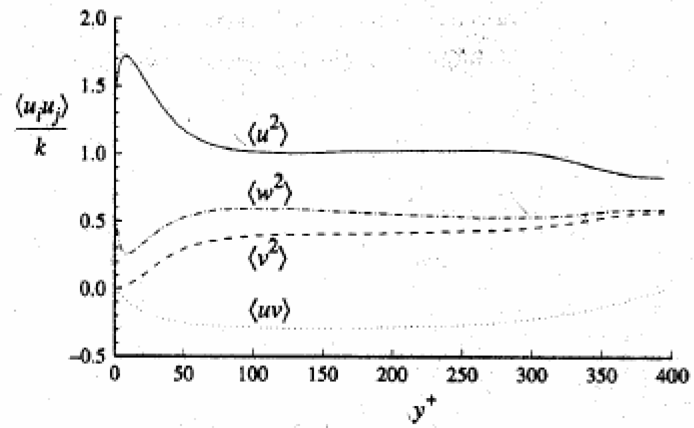

Fig. 7.15. Profiles of Reynolds stresses normalized by the turbulent kinetic energy from DNS of channel flow at Re = 13,750 (Kim *et al.* 1987).

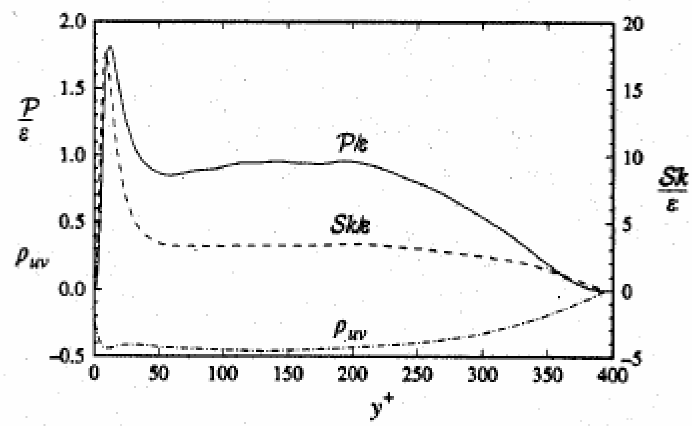

Fig. 7.16. Profiles of the ratio of production to dissipation ($\mathcal{P}/\varepsilon$), normalized mean shear rate ($\mathcal{S}k/\varepsilon$), and shear stress correlation coefficient (ρ_{uv}) from DNS of channel flow at Re = 13,750 (Kim *et al.* 1987).

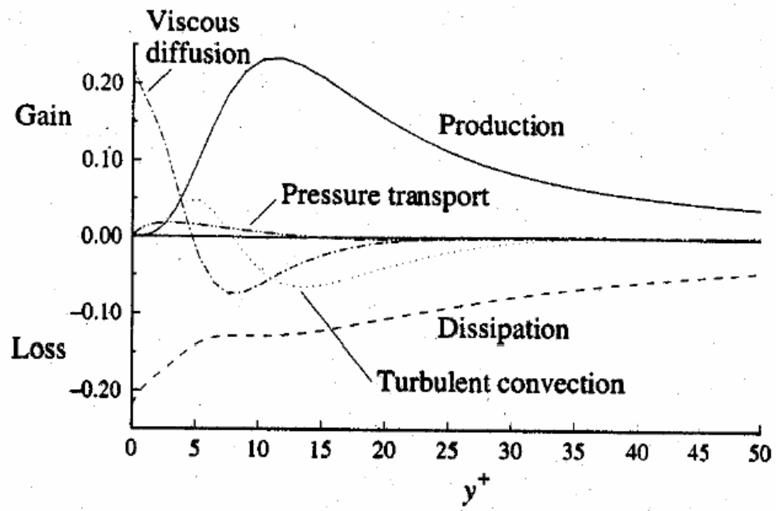

Fig. 7.18. The turbulent-kinetic-energy budget in the viscous wall region of channel flow: terms in Eq. (7.64) normalized by viscous scales. From the DNS data of Kim *et al.* (1987). Re = 13,750.