Handout: Boundary Layers

Differences to Turbulent Channel Flow

- **Boundary layer develops in the flow direction,** $\delta = \delta(x)$
- τ_w **not known a priori**
- **Outer part of the flow consists of intermittent turbulent/non-turbulent motion**

But behavior of the flow in the inner layer $\left(y/\delta(x) < 0.1\right)$ is very similar to channel flow

In the defect layer $\left(y/\delta(x) > 0.1\right)$, the departures from the log law are more significant

Definition

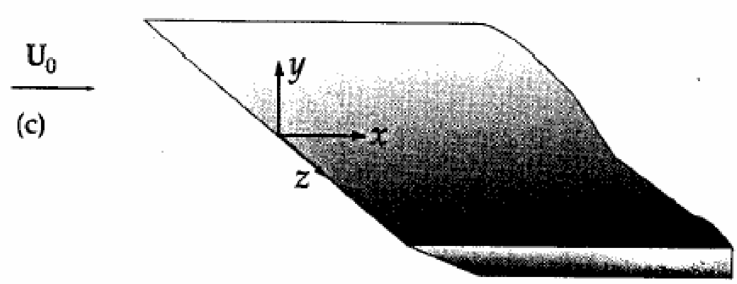

Fig. 1 Sketch of a flat-plate boundary layer

Assumptions:

- Statistics independent of z: <W>=0
- As the boundary layer continuously develops in the x direction, statistics depend on both x and y.
- $U_0=U_0(x)$

Bernoulli equation for free stream:

$$p_0(x) + \frac{1}{2}\rho U_0^2(x) = const$$
$$-\frac{\partial p_0(x)}{\partial x} = \rho U_0(x)\frac{\partial U_0(x)}{\partial x} \qquad (1)$$

Thus, accelerating flow corresponds to a negative or ‘favorable’ pressure gradient.

Decelerating flow yields a positive or ‘adverse’ pressure gradient

Boundary layer thickness $\delta(x)$ defined as the y value at which

$$\langle U \rangle (x,y)|_{y=\delta} = 0.99 U_0(x)$$

This quantity depends on small velocity differences. More reliable ways to characterize the thickness of boundary layer are:

Displacement thickness

$$\delta^*(x) \equiv \int_0^\infty (1 - \frac{\langle U \rangle}{U_0}) dy$$

Momentum thickness

$$\theta(x) \equiv \int_0^\infty \frac{\langle U \rangle}{U_0} (1 - \frac{\langle U \rangle}{U_0}) dy$$

Relevant Reynolds numbers:

$$\mathrm{Re}_x \equiv \frac{U_0 x}{\nu}, \mathrm{Re}_\delta \equiv \frac{U_0 \delta}{\nu}, \mathrm{Re}_{\delta^*} \equiv \frac{U_0 \delta^*}{\nu}, \mathrm{Re}_\theta \equiv \frac{U_0 \theta}{\nu}$$

Critical Reynolds number for zero-pressure gradient boundary layer:

Flow is laminar from x=0 to a location x (which defines the start of transition) which corresponds to

$$\mathrm{Re}_{x,crit} \approx 10^6$$

but this value is also dependent on level of disturbances in the free stream.

The boundary layer typically becomes fully turbulent over some distance (~30% of the distance from the leading edge to the start of transition)

Mean Momentum Equations

- Flow develops in the x direction
- Axial stress gradients are small compared to cross-stream gradients

The lateral momentum equation reduces to:

$$0 = -\frac{1}{\rho}\frac{\partial\langle p\rangle}{\partial y} - \frac{\partial\langle v^2\rangle}{\partial y} \qquad (2)$$

Integrate (2) from the wall (y=0) to freestream where the velocity fluctuations are equal to zero:

$$p_0(x) = p_w(x) \quad \text{(=wall pressure)}$$

Integrate (2) from 0 to y

$$\langle p\rangle + \rho\langle v^2\rangle = p_w(x) = p_0(x)$$

The Mean Axial Momentum Equation is:

$$\langle U\rangle\frac{\partial\langle U\rangle}{\partial x} + \langle V\rangle\frac{\partial\langle U\rangle}{\partial y} = -\frac{1}{\rho}\frac{\partial\langle p\rangle}{\partial x} + \nu\frac{\partial^2\langle U\rangle}{\partial y^2} - \frac{\partial\langle uv\rangle}{\partial y}$$

$$= -\frac{1}{\rho}\frac{\partial\langle p_0\rangle}{\partial x} + \frac{\partial\langle v^2\rangle}{\partial x} + \nu\frac{\partial^2\langle U\rangle}{\partial y^2} - \frac{\partial\langle uv\rangle}{\partial y}$$

and using (1)

$$= \frac{1}{\rho}\frac{\partial\tau}{\partial y} + U_0\frac{\partial U}{\partial x} \qquad (3)$$

where the shear stress is defined as:

$$\tau = \rho\nu\frac{\partial\langle U\rangle}{\partial y} - \rho\langle uv\rangle \qquad (4)$$

and $\frac{\partial\langle v^2\rangle}{\partial x}$ was neglected as were all the other contributions from streamwise gradients of Reynolds stresses (boundary layer approximation) in the original form of the axial momentum equation.

In contrast to channel flow, convective terms are non-zero and cannot be determined easily!

At the wall (y=0) use (3) and the fact that the convective terms are zero, to obtain

$$\left.\frac{\partial\tau}{\partial y}\right|_{y=0} = -\frac{1}{\rho}\frac{dp_0}{dx}$$

If the freestream pressure gradient is zero and using the definition of the shear stress (4) and the fact that <uv> increases from zero at the wall proportional to y^3:

$$\left.\frac{\partial\tau}{\partial y}\right|_{y=0} = \rho\nu\left.\frac{\partial^2\langle U\rangle}{\partial y^2}\right|_{y=0} = 0$$

which shows that the mean streamwise velocity profile varies linearly with y near y=0.

Integration of momentum equation leads to von Karman integral momentum equation

(Derivation for $\frac{\partial\langle p_0\rangle}{\partial x} = 0$ so U_0 is not a function of x)

Use continuity equation and rewrite (3) for zero pressure gradient as:

$$\frac{\partial\langle U\rangle^2}{\partial x} + \frac{\partial\langle U\rangle\langle V\rangle}{\partial y} = \frac{1}{\rho}\frac{\partial\tau}{\partial y}$$

Integrate from 0 to ∞ with y:

$$\int_0^\infty \frac{\partial\langle U\rangle^2}{\partial x}dy + \int_0^\infty \frac{\partial\langle U\rangle\langle V\rangle}{\partial y}dy = \frac{1}{\rho}\int_0^\infty \frac{\partial\tau}{\partial y}dy$$

Add/substract $\int_0^\infty \frac{\partial\langle U\rangle U_0}{\partial x}dy$ and integrate second and last terms (in the free stream <V>=0 and the shear stress τ=0)

$$\int_0^\infty \frac{\partial}{\partial x}[\langle U\rangle(U_0 - \langle U\rangle)]dy + \int_0^\infty U_0 \underbrace{\frac{\partial\langle U\rangle}{\partial x}}_{Continuity: -\frac{\partial\langle V\rangle}{\partial y}} dy + \underbrace{[\langle U\rangle\langle V\rangle]_0^\infty}_{=0} = \frac{1}{\rho}\underbrace{[\tau]_0^\infty}_{=-\tau_w}$$

Recall definition of θ

$$-U_0^2\frac{\partial\theta}{\partial x} - U_0\underbrace{\int_0^\infty \frac{\partial\langle V\rangle}{\partial y}dy}_{[\langle V\rangle]_0^\infty = 0} = -\frac{\tau_w}{\rho}$$

$$\Rightarrow \tau_w = \rho U_0^2\frac{\partial\theta}{\partial x}$$

Mean Velocity Profiles

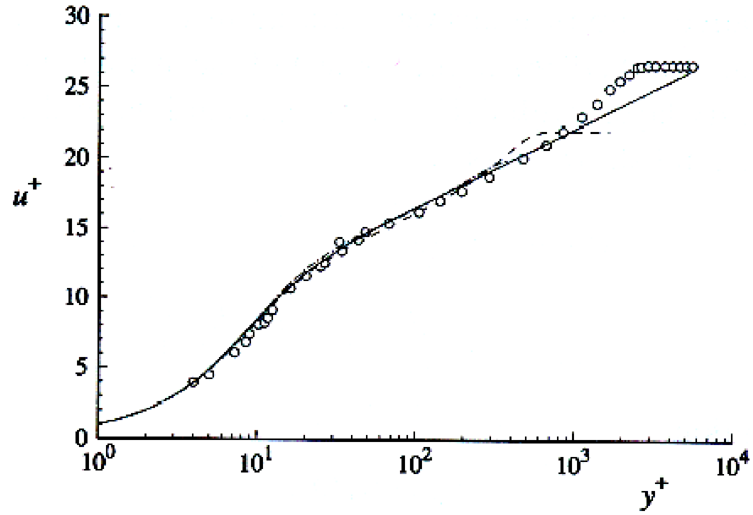

Fig. 7.27. Mean velocity profiles in wall units. Circles, boundary-layer experiments of Klebanoff (1954), $Re_\theta = 8{,}000$; dashed line, boundary-layer DNS of Spalart (1988), $Re_\theta = 1{,}410$; dot–dashed line, channel flow DNS of Kim *et al.* (1987), Re = 13,750; solid line, van Driest's law of the wall, Eqs. (7.144)–(7.145).

Fig. 2 Mean velocity profiles in wall units (experiments and simulations)

$\Rightarrow$ Law of the wall still holds in the log-law region, buffer layer and the viscous sublayer ($u^+=y^+$).

Question: What form does the law of the wall take in the buffer layer ($5<y^+<30\text{-}50$)?

Van Driest Damping function for buffer layer

Mixing length hypothesis: $-\rho < uv >= \nu_t \dfrac{\partial < U >}{\partial y} = l_m^2 \left(\dfrac{\partial < U >}{\partial y}\right)^2$

So from (4):

$$\frac{\tau(y)}{\rho} = \nu \frac{\partial \langle U \rangle}{\partial y} + \nu_t \frac{\partial \langle U \rangle}{\partial y}$$

$$= \nu \frac{\partial \langle U \rangle}{\partial y} + l_m^2 \left(\frac{\partial \langle U \rangle}{\partial y}\right)^2$$

with $l_m^+ = l_m / \delta_\nu$, $u^+ = <U> / u_\tau$, $u_\tau = \sqrt{\tau_w / \rho}$, $y^+ = y / \delta_\nu$

$$\frac{\tau}{\tau_w} = \frac{\partial u^+}{\partial y^+} + l_m^{+2} (\frac{\partial u^+}{\partial y^+})^2$$

For inner layer $\tau / \tau_w \approx 1$

$$\Rightarrow (\frac{\partial u^+}{\partial y^+})^2 + \frac{1}{l_m^{+2}} \frac{\partial u^+}{\partial y^+} - \frac{1}{l_m^{+2}} = 0$$

Solution

$$\frac{\partial u^+}{\partial y^+} = \frac{2}{1 + \sqrt{4 l_m^{+2} + 1}} \qquad (5)$$

To integrate (5) all what we have to do is to specify $l_m^+ = l_m^+ (y^+)$

But we know that in the Log Layer: l_m=ky

$$\Rightarrow l_m^+ = \kappa y^+ \qquad (6)$$

which can be used to determine u^+ in the log-layer.

If same specification of the mixing length would be used in the viscous sub-layer:

$\Rightarrow \langle uv \rangle = -\nu_t \frac{\partial \langle U \rangle}{\partial y} = -(ky \quad)^2 \frac{\partial \langle U \rangle}{\partial y} \sim (y)^2$ whereas it is well known that

<uv>~$(y)^3$

$\Rightarrow$ incorrect y^+ (or y) dependence (should be y^{+3}):

So the specification l_m=ky should be reduced, or damped, near the wall.

$\Rightarrow$ Van Driest damping function assures proper transition to viscous sublayer

$$l_m^+ = \kappa y^+ [1 - \exp(-\frac{y^+}{A^+})] \quad \text{with } A^+=26 \qquad (7)$$

This is a purely empirical formula, but it works reasonably well and it is used in many wall models, especially in LES. For large y^+, the damping function tends to unity and the log law is recovered.

Equation (7) can be used to integrate equation (5) over both the viscous sublayer and the log layer to determine u^+ and thus the mean velocity profile <U>.

Velocity Defect Law

In the defect layer (y/δ>0.2, say) the mean velocity deviates from the log law as can be seen from Fig. 3.

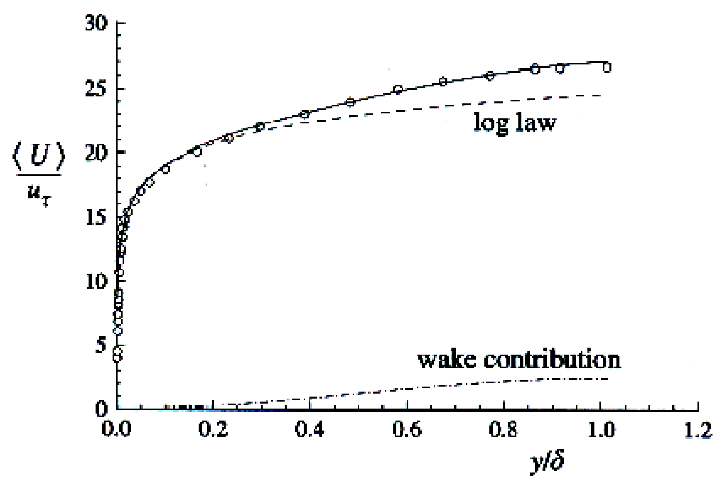

Fig. 7.28. The mean velocity profile in a turbulent boundary layer showing the law of the wake. Symbols, experimental data of Klebanoff (1954); dashed line, log law ($\kappa = 0.41, B = 5.2$); dot–dashed line, wake contribution $\Pi w(y/\delta)/\kappa$ ($\Pi = 0.5$); solid line, sum of log law and wake contribution (Eq. (7.148)).

Fig. 3 Mean velocity profile in a turbulent boundary layer

From an extensive examination of boundary-layer data, Coles (1956) showed that the mean velocity profile over the whole boundary layer is well predicted by the sum of two functions:

$$\frac{\langle U \rangle}{u_\tau} = \underbrace{f_w\left(\frac{y}{\delta_\nu}\right)}_{Law\ of\ the\ wall} + \underbrace{\frac{\Pi}{\kappa} w\left(\frac{y}{\delta}\right)}_{Law\ of\ the\ wake}$$

The wake function $w(y/\delta)$ is assumed to be universal (same for all boundary layers), and is defined to satisfy the normalization conditions

$$w(0)=1 \text{ and } w(1)=2$$

The wake strength parameter Π is flow dependent
A convenient approximation for $w(y/\delta)$ is:

$$w\left(\frac{y}{\delta}\right) = 2\sin^2\left(\frac{\pi}{2}\frac{y}{\delta}\right)$$

Approximate f_w by the log law

$$\frac{\langle U\rangle}{u_\tau} = \frac{1}{\kappa}\ln\left(\frac{y}{\delta_\nu}\right) + B + \frac{\Pi}{\kappa}w\left(\frac{y}{\delta}\right) \qquad (8)$$

For $y=\delta$

$$\frac{U_0}{u_\tau} = \frac{1}{\kappa}\ln\left(\frac{\delta}{\delta_\nu}\right) + B + \frac{2\Pi}{\kappa} \qquad (9)$$

$$= \frac{1}{\kappa}\ln\left(\mathrm{Re}_\delta \frac{u_\tau}{U_0}\right) + B + \frac{2\Pi}{\kappa} \qquad (10)$$

⇒ For given Re_δ this equation can be solved for u_τ/U_0
⇒ Skin friction coefficient is

$$c_f = \frac{\tau_w}{1/2\rho U_0^2} = 2\left(\frac{u_\tau}{U_0}\right)^2$$

Velocity defect law (subtract (8) from (9))

$$\frac{U_0 - \langle U\rangle}{u_\tau} = \frac{1}{\kappa}\left[-\ln\left(\frac{y}{\delta}\right) + \Pi\left(2 - w\left(\frac{y}{\delta}\right)\right)\right]$$

Eddy Viscosity in Defect Layer

Eddy viscosity definition
$$\nu_t = \frac{\tau}{\rho \partial \langle U \rangle / \partial y}$$

Eddy viscosity model (mixing length)
$$\nu_t = l_m^2 \frac{\partial \langle U \rangle}{\partial y}$$

Defect layer: the shear stress $\tau(y)$ is less than τ_w and the velocity gradient $\frac{\partial \langle U \rangle}{\partial y}$ is larger than the value $u_\tau / (\kappa y)$ given by the log law.

This means that the value of ν_t is less than the one given by the log-law formula $\nu_t = u_\tau \kappa y$ and, consequently, the mixing length l_m is smaller than κy in the defect later. This is confirmed by results from DNS in Fig. 4

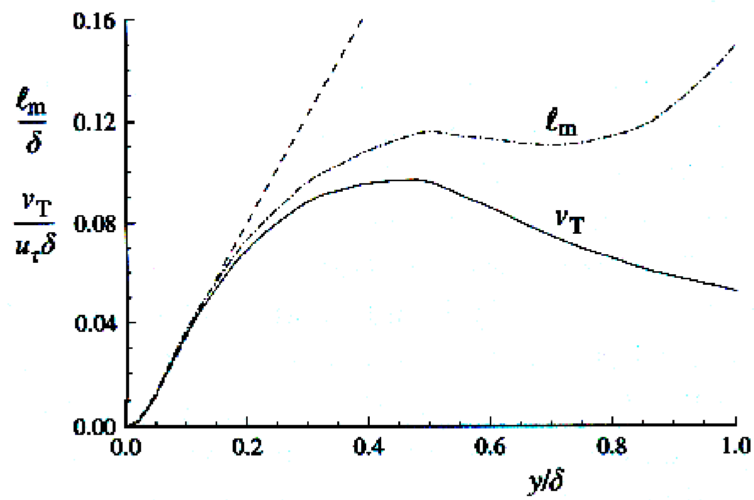

Fig. 7.30. Turbulent viscosity and mixing length deduced from direct numerical simulations of a turbulent boundary layer (Spalart 1988). Solid line, ν_T from DNS; dot–dashed line, ℓ_m from DNS; dashed line ℓ_m and ν_T according to van Driest's specification (Eq. (7.145)).

Fig. 4 Turbulent viscosity and mixing length variation in a turbulent boundary layer.

Thus l_m has to be adjusted. One simple way to do that is

$$l_m = \min(\kappa y, 0.09\delta)$$