058:268 Turbulent Flows 2004
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HOMEWORKS:

<u>Assignment I</u> - 01/26/04, Due 02/04/04

1) A cubical box of volume L^3 is filled with fluid in turbulent motion. No source of energy is present, so that the turbulence decays. Because the turbulence is confined to the box, its length scale may be assumed to be equal to L at all times. Derive an expression for the decay of the kinetic energy $3u^2/2$ as a function of time. As the turbulence decays, its Reynolds number decreases. If Re=uL/ν becomes smaller than 10 say, the inviscid estimate $\varepsilon = u^3 / L$ should be replaced by an estimate of the type $\varepsilon = c \nu u^2 / L$, because the weak eddies remaining at low Reynolds number lose their energy directly to viscous dissipation. Compute the coefficient c required for the dissipation rate to be continuous at Re=10. Derive an expression for the decay of the kinetic energy when Re<10 (this is called the final period of decay). If L=1m, $\nu=15*10^{-6}$ m2/s and u=1m/s at time t=0, how long does it take before the turbulence enters the final period of decay? Assume that the effects of the walls of the box on the decay of the turbulence may be ignored.

2) For a two-dimensional steady turbulent flow in Cartesian coordinates (x,y), with mean velocity components (U,V), write

(a) the continuity and momentum equations;

(b) the mean internal energy equation;

(c) the mean vorticity equation;

(d) the turbulent kinetic energy equation;

Simplify these for fully-developed flow between two parallel plates, a distance 2h apart, under the influence of a piezometric pressure gradient dp/dx. Consider the boundary conditions that apply in this case.

3) TASK 1

Assignment II - 02/04/04, Due 02/16/04

1) Derive the vorticity equation 2.11 from the Navier Stokes equations. Hence derive the mean vorticity equation (2.28). With this expressed in Cartesian coordinates, discuss the connection between the Reynolds stresses and secondary flow in steady-fully developed flow in a straight duct.
Reference: Su and Friedrich (1994), ASME j. Fluids Engrg., 116, pp. 677-684.

2) Derive the Poisson equation for the Reynolds averaged pressure field
Show that $-\frac{1}{\rho}\nabla^2\overline{P} = \frac{\partial \overline{U_i}}{\partial x_j}\frac{\partial \overline{U_j}}{\partial x_i} + \frac{\partial^2 \overline{u_i u_j}}{\partial x_i \partial x_j}$ where $U_i = \overline{U_i} + u_i$, etc.

3) Denote the average of a quantity by <>. Let $\phi(x,t)$ be a conserved scalar quantity that satisfies the conservation equation:

$$\frac{\partial \phi}{\partial t} + \nabla \cdot (\vec{U}\phi) = \Gamma \nabla^2 \phi$$

It is very easy to show that the Reynolds averaged equation for this scalar is given by

$$\frac{\partial <\phi>}{\partial t} + <\vec{U}> \nabla \cdot <\phi> = \Gamma \nabla^2 <\phi> - \nabla \cdot (<\vec{u}\phi'>) \text{ where } \phi = <\phi> + \phi',\ \vec{U} = <\vec{U}> + \vec{u}$$

a) Show that the transport equation for the scalar variance is

$$\frac{\partial <\phi'^2>}{\partial t} + <\vec{U}> \nabla \cdot <\phi'^2> = \Gamma \nabla^2 <\phi'^2> - \nabla \cdot (<\vec{u}\phi'^2>) - 2 <\vec{u}\phi'> \cdot \nabla <\phi> - 2\Gamma <\nabla\phi' \cdot \nabla\phi'>$$

b) Identify the production term, the dissipation term, the turbulent scalar flux term and the viscous transport term
c) Introduce the gradient diffusion model $<\vec{u}\phi'> = -\Gamma_t \nabla <\phi>$ into the scalar variance equation

d) Assuming there is a balance between production and dissipation in the scalar variance equation, and the eddy diffusivity is known, provide a model for the scalar dissipation rate ε_ϕ (which is one of the terms in the equation for the scalar variance)

e) In the k-ε model, transport equations are solved for k and ε, so that these are known quantities. Assuming the time scale of the turbulence decay k/ε is equal to the time scale of the scalar variance decay defined analogously, and using the model for ε_ϕ, provide a model for the scalar variance.

4. Derive the equation for the fluctuating velocity: $\partial_t u_i + U_k \partial_k u_i + u_k \partial_k U_i + \partial_k (u_k u_i - \overline{u_k u_i}) = -(1/\rho)\partial_i p + \nu \nabla^2 u_i$

and then the Reynolds stress transport equation:

$$\partial_t \overline{u_i u_j} + U_k \partial_k \overline{u_i u_j} = -\partial_k \overline{u_k u_i u_j} - (1/\rho)(\overline{u_j \partial_i p} + \overline{u_i \partial_j p}) - 2\nu \overline{\partial_k u_i \partial_k u_j} + \nu \nabla^2 \overline{u_i u_j} - \overline{u_j u_k} \partial_k U_i - \overline{u_i u_k} \partial_k U_j$$

starting from the momentum & continuity equations for the total velocity:

$\partial_t (U+u)_i + (U+u)_k \partial_k (U+u)_i = -(1/\rho)\partial_i (P+p) + \nu \nabla^2 (U+u)_i$

$\partial_i (U+u)_i = 0$, $\overline{U} = U$, $\overline{u} = 0$

Symbolically the steps are

$$RS_{ij} = \overline{u_j \left[NS(U+u)_i - \overline{NS(U+u)_i} \right]} + \overline{u_i \left[NS(U+u)_j - \overline{NS(U+u)_j} \right]}$$

(you are basically taken the first moment of the momentum equation)
How many independent equations and how many unknowns are there? Why not also form the moment of the continuity equation?

5. Using the N-S equations:

$$\partial_t U_i + (U)_k \partial_k U_i = -(1/\rho)\partial_i P + \nu \nabla^2 U_i - \partial_j \overline{u_i u_j}$$

show that the rate at which the mean energy (per unit mass) $1/2 U_i U_i$ is lost to turbulence is $\overline{u_j u_i} \partial_i U_j = -PROD$ +viscous_contribution. Hint: Terms that are conservative (can be written as divergence of a function) are just redistributing energy, so they are not an 'energy loss'. What this exercise demonstrates is that the term 'production' is actually referring to the transfer of energy from the mean flow to the turulence, and not to a net source of energy.

6. Anisotropy equation: The Reynolds stress anisotropy tensor is defined as $b_{ij} = \overline{u_i u_j}/k - 2/3\delta_{ij}$. Using the fact that the equation for the tke in homogeneous turbulence is $\partial_t k = P - \varepsilon$ (P is the production term) the equations for the turbulent stresses $\partial_t \overline{u_i u_j} = -\phi_{ij} + P_{ij} - \varepsilon_{ij}$ where ϕ_{ij} is the pressure velocity correlation (or redistribution) term, derive the following equation for b_{ij}:

$$\partial_t b_{ij} = -\frac{\phi_{ij} + \varepsilon_{ij}}{k} - \frac{2}{3}(\partial_j U_i + \partial_i U_j) - b_{ik}\partial_j U_k - b_{jk}\partial_i U_k - (b_{ij} + 2/3\delta_{ij})(P - \varepsilon)/\mathrm{k}$$

where P/k= $-b_{ki}\partial_k U_i$. For the other definitions please see pag 52-57 in Durbin's book.

Assignment III - 02/16/04, Due 02/23/04

1. Find the probability distribution function, P(u), and the probability density function, β(u), for the following periodic functions with period T = 2:

   (a) The triangle wave: $u(t) = t \quad 0 < t < 1$
   $u(t) = 2 - t \quad 1 < t < 2$

   (b) The sine wave: $u(t) = \sin \pi t$ (with T = 2)

2. Analysis of an ensemble of experimental data for a random variable u(t) reveals that a good approximation for its probability density β(u) is

$$\beta(u) = 0.1\{1 + \cos[0.628(u - 20)]\} \quad 15<u<25$$

   (a) Plot this function
   (b) Calculate the mean and standard deviation for u
   (c) Determine the Gaussian probability distribution for a variable with the same first and second moments as in (b) above. Plot the Gaussian probability density on the same plot as (a), and comment on the comparison.

3. Determine the autocorrelation function ρ(τ) for the periodic signal u(t) = sin ωt using the time average. Repeat for u(t) = cos ωt, and comment on the results.

4. Show that, for a stationary random function u(t), with autocorrelation coefficient ρ(τ),

   (a) $$\overline{\left[\int_a^b u(t)\,dt\right]^2} = 2\,\overline{u^2}\int_0^{b-a} \rho(\tau)\,(b - a - \tau)\,d\tau$$

   (b) $$\overline{\left(\frac{du(t)}{dt}\right)^2} = -\,\overline{u^2}\left[\frac{d^2\rho(\tau)}{d\tau^2}\right]_{\tau = 0}$$

5. What is the Fourier transform of the delta function, δ(t)?

6. In a spray, the probability density function of droplet diameter D is uniform in the range 50<D<100 μm and zero otherwise. Determine the probabilty density function β(D) and the Sauter mean diameter, defined as $\overline{D^3}/\overline{D^2}$.

7. Consider a round oil jet injected into still water. The two fuids are perfectly immiscible (i.e., they do not mix at the molecular level but can be broken down into small oil drops in water, and vice versa, much like salad oil). The oil has a density of 600 kg/m3 and the density of water is 1000 kg/m3. Let f represent the mass fraction of oil at any location.

   (a) Find an expression for the probability density function b(f) as a function of $\bar{f}$ (hint: since the two fluids do not mix, b(f) comprises two delta functions).

   (b) Calculate the mean density $\bar{\rho}$ as a function of $\bar{f}$. Recall that $\bar{\rho} = \int_0^1 \rho(f)\beta(f)df$

8. Most computer libraries have a random number algorithm generator that generates values between 0 and 1 with equal probability i.e. P($\tilde{u}$)=1 for 0<$\tilde{u}$<1. Show that the mean $\bar{\tilde{u}}$ =1/2 and that $\overline{u^2} = 1/12$, where u is the fluctuation u=$\tilde{u}$ - $\bar{\tilde{u}}$. Deduce that $\xi = \sqrt{12}(\tilde{u} - 1/2)$ has $\bar{\xi} = 0$ and $\overline{\xi^2} = 1$ (you have to proof that analytically first). A Gaussian random variable can be approximated by summing N values ξ_i and normalizing by $\sqrt{N}$. E.g., for N=16, $\xi_G = 1/4\sum_1^{16}\xi_i$. Program this and verify by averaging a large number (10,000 say) of values that $\bar{\xi}_G = 0$ and $\overline{\xi_G{}^2} = 1$.

Assignment IV - 02/23/04, Due 03/03/04

1. Show that, in isotropic turbulence,

   (a) $\overline{p'v'_i} = 0$

   (b) $\overline{\left(\frac{\partial u}{\partial x}\right)^2} = \frac{1}{2}\overline{\left(\frac{\partial v}{\partial x}\right)^2}$, etc.

2. For homogeneous isotropic turbulence the two point correlation $R_{ij}(\vec{x}+\vec{r},t) = < u_i(\vec{x}+\vec{r},t)u_j(\vec{x},t) >$ can be written as $R_{ij} = u'^2\left[(f-g)\frac{r_i r_j}{r^2} + g\delta_{ij}\right]$ where <> is the average operator.

   Show first that $\frac{\partial R_{ij}}{\partial r_j} = 0$. Then using this show that $g = f + \frac{r}{2}f'$

3. Assuming that the turbulence is isotropic, estimate the value of the microscale λ corresponding to a decrease in rms turbulence intensity in a body of water from 1 to 0.5 m/s in a time of 20 seconds.

4. The rms diameter of a cloud of marked particles in a field of isotropic turbulence is found to grow initially at a rate of 0.03 m/s. If the Lagrangian time scale is 10 s, what will be the rms diameter of the cloud at t = 60 s?

5. For homogeneous turbulence, show that the equation for the turbulent kinetic energy reduces to

$$\frac{\partial k}{\partial t} = < u_i u_j > \frac{\partial < U_j >}{\partial x_i} - \varepsilon$$

Recall that in homogeneous turbulence fluctuations are not a functions of space while the mean velocity gradient in each direction can assume a constant non-zero value. From the definition of ε show exactly what terms are not equal to zero. For homogeneous isotropic turbulence, show that the equation for the turbulence kinetic energy reduces to

$$\frac{\partial k}{\partial t} = -\varepsilon$$

Recall that in isotropic homogeneous turbulence the mean flow field is shear free.

Assignment V - 03/03/04, Due 03/08/04

1. Consider the turbulent mixing layer between two streams of velocity U_1 and U_2. Show that the layer thickness grows linearly with distance from the origin. Describe how you would determine the velocity distribution across the mixing layer.

2. Investigate a similarity solution for the axisymmetric wake, in laminar and turbulent flow, to determine the growth and decay laws, and the velocity profiles. In the turbulent case, assume a constant eddy viscosity across the wake.

Assignment VI - 03/03/04, Due 03/08/04

1) Consider vertical axisymmetric flow, e.g., a thermal plume from a point heat source, or a boundary layer outside a vertical cylinder. With buoyancy present, the equations of motion (under thin shear-layer approximations) in laminar flow may be written

$$\frac{\partial(ru)}{\partial x}+\frac{\partial(rv)}{\partial r}=0 \tag{1}$$

$$u\frac{\partial(u)}{\partial x}+v\frac{\partial(u)}{\partial r}=\beta(t-t_\infty)+\frac{\nu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) \tag{2}$$

$$u\frac{\partial(t)}{\partial x}+v\frac{\partial(t)}{\partial r}=\frac{\alpha}{r}\frac{\partial}{\partial r}\left(r\frac{\partial t}{\partial r}\right) \tag{3}$$

where t is temperature, t_∞ is the ambient temperature, β is the coefficient of thermal expansion, and α is the thermal diffusivity.

(i) Investigate similarity solutions for a laminar axisymmetric thermal plume to determine its growth and decay laws. To do this, you need to consider the integral constraints, introduce appropriate velocity and length scales, and seek conditions which would yield similarity solutions. You will find that it is useful to introduce a stream function, and define a dimensionless temperature difference: q = $(t - t_\infty)/(t_o - t_\infty)$, where t_o is the temperature at the plume axis. Obtain the ordinary differential equations that govern the velocity and temperature profiles (do not solve unless you want to make a project out of it).

(ii) Next consider turbulent flow. A two-dimensional turbulent plume is discussed in Tennekes and Lumley, sec. 4.6, pp. 135-144. Develop the equations for the axisymmetric turbulent plume. Then investigate similarity solutions to determine the growth and decay laws. Again, obtain the ordinary differential equations that govern the velocity and temperature profiles using either eddy-viscosity or mixing-length hypothesis.

2) The momentum equation for the temporally evolving turbulent mixing layer is given by

$$\frac{\partial <U>}{\partial t}=-\frac{\partial <uv>}{\partial y}$$

a) Introduce the gradient transport model, assuming constant eddy viscosity
b) Show that by introducing the similarity coordinate

$$\eta = \frac{y}{2\sqrt{\nu_t t}}$$

and the similarity variable

$$f = \frac{< U >}{U_s}$$

where $U_s = U_h - U_l$ and U_l and U_h are the velocity at $y \longrightarrow \infty$ and $y \longrightarrow -\infty$, the momentum equation can be written as

$$-2\eta \frac{df}{d\eta} = \frac{\partial^2 f}{d\eta^2}$$

c) Show that <U> is given by

$$< U(\eta) >= \frac{1}{2} U_s erf(\eta) + U_c$$

Assignment VII - Due 03/22/04

TASKS 2,3,4,5.

Assignment VIII - Due 04/05/04

TASKS 6,7.

Assignment IX - 03/08/04, Due 04/21/04

1. The velocity distribution in turbulent flow in a pipe of radius a may be approximated by a power-law

$$\frac{u}{U} = \left[\frac{y}{a}\right]^{1/m}$$

where y is measured from the wall. The power depends on the Reynolds number. The momentum equation shows that the pressure varies linearly along the pipe, and the total stress (molecular plus turbulent) varies linearly across the pipe. Use this information to find an expression for the distribution of the mixing length across the pipe. Plot this in an appropriate non-dimensional form for m = 7.

2. An experiment is performed on fully turbulent channel flow at $\mathrm{Re} = \frac{2\delta\overline{U}}{\nu} = 10^5$ The fluid is water ($\nu = 1.14*10^{-6} m/s$) and the channel half size $\delta = 2cm$. The skin friction coefficient is found to be $C_f = 4.4*10^{-3}$. Determine: $\overline{U}, u_\tau/\overline{U}, \mathrm{Re}_\tau$ and δ_v/δ where $\delta_v = \nu/u_\tau$ is the viscous lengthscale. What are the thicknesses of the viscous wall region ($y^+<50$) and of the viscous sublayer, both as fractions of δ and in millimeters?

3. Carry out a series expansion of the instantaneous velocity components to determine the behavior of the Reynolds stress components, turbulent kinetic energy, and dissipation rate near a free surface (zero stress boundary). Comment on the differences between this and a solid wall.

4. Simplify the k-ε, k-τ and k-ω models for the logarithmic layer (neglect convection, diffusion, and viscous terms). Determine the value of the Karman constant, κ, implied by these models and compare with the experimental value of 0.418. Hint: for all models, assume a solution of the form:

$$\frac{dU}{dy} = \frac{U_\tau}{\kappa y}, \qquad k = \frac{U_\tau^2}{\sqrt{C_\mu}}, \qquad \nu_\tau = \kappa U_\tau y$$

5. Beginning with the k-ω model of Wilcox and σ* = σ, make the formal change of variables ε = β*ωk and derive the implied k-ε model. Express your final results in

standard k-ε model notation and determine the implied values of the closure coefficients C_μ , $C_{\varepsilon 1}$, $C_{\varepsilon 2}$, σ_k and σ_ε in terms of α, β, β*, σ and σ*.

6. Show that the expansion for the Reynolds shear stress at the wall can be written as:

$$< uv > / u_\tau^2 = -\sigma {y^+}^3$$

where the nondimensional coefficient σ may be assumed to be independent of the Reynolds number. For the flow between two walls (fully developed channel flow) show using the momentum equations (remember that one can show that the total stress varies linearly across the channel) that in wall coordinates the expansion for the streamwise velocity is:

$u^+ = y^+ - \dfrac{{y^+}^2}{2\,\mathrm{Re}_\tau} - \dfrac{1}{4}\sigma {y^+}^4$. Thus if $\mathrm{Re}_\tau >> 1$ the expansion for the law of the wall is

$$u^+ = y^+ - \frac{1}{4}\sigma {y^+}^4$$

7. Mixing length model with Van Driest damping: In a wall boundary layer, the log-law applies in the region 35<y^+<0.2 δ_{99}^+ (δ_{99}^+ is the thickness of the b.l. in wall units). To extend the mixing length model all the way to y=0, Van Driest suggested that κy should be multiplied by an 'exponential damping function'. Thus the turbulence length scale near the wall becomes $l_m = \kappa y(1 - e^{-y^+/A^+})$, where the Von Karman constant is 0.41. Apply this to the constant total stress layer, $-\overline{uv} + \nu \partial_y U = u_\tau^2$ to find a formula for $\partial_y U$. Integrate this numerically and make a log-linear plot of U^+ vs. y^+. What is the additive constant B that you obtain from the log law if A^+=26? What value of A^+ gives B=5.5?

Hint: The formula you have to obtain is $\frac{dU^+}{dy^+} = \frac{\sqrt{1+4{l_m^+}^2}-1}{2{l_m^+}^2}$

8. A shortcoming of the k-ε model: Show that in incompressible flow the eddy viscosity formula $-\overline{u_i u_j} = 2\nu_t S_{ij} - 2/3k\delta_{ij}$ gives the same rate of turbulent energy production as $P = 2\nu_t S_{ij} S_{ij}$ irrespective of the mean rate of rotation tensor $\Omega_{ij} = \frac{1}{2}(\partial_i U_j - \partial_j U_i)$. When turbulence is rotated (e.g., the case of the flow in turbomachinery) the centrifugal acceleration can affect the turbulent energy. Discuss whether the classical k-ε model can predict such effects. There is an analogy between rotation and streamline curvature, so your conclusion should apply to effects of curvature on the turbulence, too.

Assignment X - Due 04/28/04

TASKS 8,9,10,11

Assignment XI - Due 05/05/04

TASKS 12,13